A lower bound for Torelli-K-quasiconformal homogeneity

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Abstract

A closed hyperbolic Riemann surface M is said to be K-quasiconformally homogeneous if there exists a transitive family $\mathcal{F}$ of K-quasiconformal homeomorphisms. Further, if all $[f] \subset \mathcal{F}$ act trivially on $H_1(M;\mathbb{Z})$, we say M is Torelli-K-quasiconformally homogeneous. We prove the existence of a uniform lower bound on K for Torelli-K-quasiconformally homogeneous Riemann surfaces. This is a special case of the open problem of the existence of a lower bound on K for (in general non-Torelli) K-quasiconformally homogeneous Riemann surfaces.

1 Introduction

K-Quasiconformal homeomorphisms of a Riemann surface M generalize the notion of conformal maps by bounding the dilatation at any point of M by $K < \infty$. Let $\mathcal{F}$ be the family of all K-quasiconformal homeomorphisms of M. If for any points $p, q \in M$, there is a map $f \in \mathcal{F}$ such that $f(p) = q$, that is, the family $\mathcal{F}$ is transitive, then M is said to be K-quasiconformally homogeneous. Quasiconformal homogeneity was first studied by Gehring and Palka in [4] in 1976 for genus zero surfaces and analogous higher dimensional manifolds. Gehring and Palka also showed that the only 1-quasiconformally homogeneous (i.e. $\mathcal{F}$ is transitive with all maps conformal) genus zero surfaces are non-hyperbolic. It was also found that there do exist genus zero surfaces which are K-quasiconformal for $1 < K < \infty$.

A more recent question is whether there exists a uniform lower bound on K for K-quasiconformally homogeneous hyperbolic manifolds. Using Sullivan's Rigidity Theorem, Bonfert-Taylor, Canary, Martin, and Taylor

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showed in [9] that for dimension $n \geq 3$, there exists such a universal constant $\mathcal{K}_n > 1$ such that for any K-quasiconformally homogeneous hyperbolic n-manifold other than $\mathbb{D}^n$, we have $K \geq \mathcal{K}_n$. In [9] it is also shown that K-quasiconformally homogeneous hyperbolic n-manifolds for $n \geq 3$ are precisely the regular covers of closed hyperbolic orbifolds. For two-dimensional surfaces, such a classification is shown to be false with the construction of K-quasiconformally homogeneous surfaces which are not quasiconformal deformations of regular covers of closed orbifolds in [13].

In dimension two, Bonfert-Taylor, Bridgeman, Canary, and Taylor showed the existence of such a bound for a specific class of closed hyperbolic surfaces which satisfy a fixed-point condition in [10]. Bonfert-Taylor, Martin, Reid, and Taylor showed in [14] the existence of a similar bound $\mathcal{K}_c > 1$ such that if $M \neq \mathbb{H}^2$ is a K-strongly quasiconformally homogeneous hyperbolic surface, that is, each member of the transitive family of K-quasiconformally homogeneous maps is homotopic to a conformal automorphism of M, then $K \geq \mathcal{K}_c$. Kwakkel and Markovic proved the conjecture of Gehring and Palka for genus zero surfaces by showing the existence of a lower bound on K for hyperbolic genus zero surfaces other than $\mathbb{D}^2$ in [1]. Additionally, it was shown by Kwakkel and Markovic that for surfaces of positive genus, only maximal surfaces can be K-quasiconformally homogeneous (Proposition 2.6 of [1]).

Here, we consider a special case of the problem for closed hyperbolic Riemann surfaces of arbitrary genus. Recall that the mapping class group of a surface M, denoted $\mathrm{MCG}(M)$, consists of homotopy classes of orientation-preserving homeomorphisms of M. In general, K-quasiconformal homeomorphisms are representatives of any mapping class in $\mathrm{MCG}(M)$. The Torelli subgroup $\mathcal{I}(M) \leq \mathrm{MCG}(M)$ contains those elements of $\mathrm{MCG}(M)$ which act trivially on $H_1(M;\mathbb{Z})$, the first homology group of M (see e.g. §2.1 and §7.3 of [3]). This means that the image of any closed curve $c \subset M$ under a Torelli map must be some curve homologous to c. We define a closed Riemann surface M to be *Torelli-K-quasiconformally homogeneous* if it is K-quasiconformally homogeneous and there exists transitive family $\mathcal{F}$ of K-quasiconformal homeomorphisms which consists of maps whose homotopy classes are in the Torelli group of M. That is, all $f \in \mathcal{F}$ are homologically trivial. Farb, Leninger, and Margalit found bounds on the dilatation of related pseudo-Anosov maps on a Riemann surface in [2]. In particular, they give the following result as Proposition 2.6:

Proposition 1.1. *If $g \geq 2$, then $L(\mathcal{I}(M)) > .197$, where L is the logarithm of the minimal dilatation of pseudo-Anosov maps in $\mathcal{I}(M)$.*

This result relies on several lemmas for pseudo-Anosov maps that can also be applied to Torelli-K-quasiconformal maps once we show that appropriate K-quasiconformal maps must exist. After proving a proposition that allows us to avoid the assumption of pseudo-Anosov maps, we will use an argument similar to that in the proof of Proposition 1.1 to give the following result on Torelli-K-quasiconformally homogeneous surfaces:

Theorem 1.2. *There exists a universal constant $K_T > 1$ such that if M is a Torelli-K-quasiconformally homogeneous closed hyperbolic Riemann surface, then $K \geq K_T$.*

A related proof of the result has very recently been given by Vlamis, along with similar results for other subgroups of $\mathrm{MCG}(M)$ in [12]. It would be interesting to find an actual value for K_T, and perhaps exhibit Torell-K-quasiconformally homogeneous surfaces with minimal K. In addition to bettering our estimate for K_T, proving the existence of a bound on K for the general case of non-Torelli maps is of course still an important open question. Should it be found that such a bound must exist, another interesting result may be a comparison between the value of this bound and our bound K_T. We may also wish to seek more specific details on which types of surfaces can be Torelli-K-quasiconformally homogeneous with small values of K, and if we can obtain stricter bounds for different types of surfaces.

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2 Preliminary Notions

First, we will introduce some relevant concepts, definitions, and lemmas which will be used throughout the paper.

2.1 Definitions

Let M be a closed hyperbolic Riemann surface of genus $g \geq 0$. Let c be a shortest geodesic on M. The injectivity radius $\iota(M)$ is the infimum over all $p \in M$ of the largest radius for which the exponential map at p is injective (see §2.1 of [1]). In particular, $|c| \geq 2\iota(M)$, where $|c|$ denotes the length of c.

Let $\gamma \subset M$ be a closed curve. We denote by $[\gamma]$ its homotopy class. Recall that in any homotopy class, there exists a unique geodesic whose length bounds the length of all elements of $[\gamma]$ from below (see e.g. [11]). We define the geometric intersection number of two closed curves $a, b \subset M$ by

$$i(a, b) = \min \#\{\gamma \cap \gamma'\}, \tag{1}$$

where the minimum is taken over all closed curves $\gamma, \gamma' \subset M$ with $[\gamma] = a$ and $[\gamma'] = b$ (as defined in [1]). From the discussion on page 804 of [2], the intersection number of a pair of homologous curves must be even.

Next, let X and Y be complete metric spaces. A map $f : X \to Y$ is said to be a *K-quasi-isometry* if for some $R > 0$ we have

$$R + Kd(x, x') \geq d(f(x), f(x')) \geq \frac{d(x, x')}{K} - R$$

for all $x, x' \in X$ (see [8]). A *quasi-geodesic* in a metric space X is a quasi-isometric map

$$\gamma : [a, b] \to X.$$

It is known that the image of a geodesic under a quasiconformal homeomorphism is a quasi-geodesic (Theorem 5.1 of [6]), and that a quasi-geodesic is within a bounded distance of a unique hyperbolic geodesic.

2.2 Previous Results

Let M be as above, and recall that the Torelli group of M is denoted $\mathcal{I}(M)$. First, we have Lemmas 2.2, 2.3, and 2.4 respectively from [2]. These will be instrumental in our final proof in Section 4.

Lemma 2.1. *Suppose that $[f] \in \mathcal{I}(M)$, that c is a separating curve, and that $[f(c)] \neq [c]$. Then $i(f(c), c) \geq 4$.*

Lemma 2.2. *Suppose that $[f] \in \mathcal{I}(M)$, that c is a nonseparating curve, and that $[f(c)] \neq [c]$. Then at least one of $i(f(c), c)$ and $i(f^2(c), c)$ is at least 2.*

Lemma 2.3. *Suppose c and c' are homologous nonseparating curves with $i(c, c') = 2$. Suppose that d, d', e, and e' are the boundary components of a 4-holed sphere as shown in Figure 1. Then d and d' are separating in M, and $[e] = -[e] = [c] = [c']$ in $H_1(M; \mathbb{Z})$.*

Next, we have Lemmas 2.2 and 2.3 from [1], respectively:

Lemma 2.4. *Let M be a K-quasiconformally homogeneous hyperbolic surface and $\iota(M)$ its injectivity radius. Then $\iota(M)$ is uniformly bounded from below (for K bounded from above) and $\iota(M) \to \infty$ for $K \to 1$.*

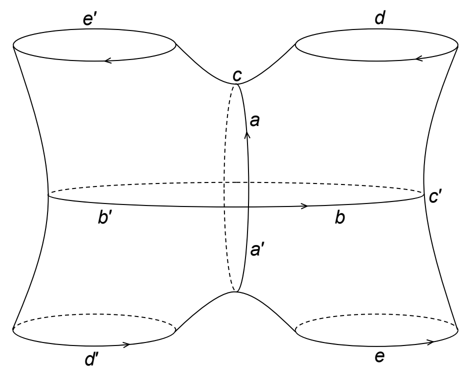

Figure 1: Illustration of Lemma 2.3, with c and c' homologous with intersection number 2. The labels a, b, a', and b' are used in the proof of Theorem 1.2. (Adapted from [2])

In particular, if c is the shortest curve on M, then $|c| \to \infty$ as $K \to 1$.

Lemma 2.5. *Let γ a simple closed geodesic in M. Let $f : M \to M$ be a K-quasiconformal homeomorphism and γ' the simple closed geodesic homotopic to $f(\gamma)$. Then $\frac{1}{K}|\gamma| \leq |\gamma'| \leq K|\gamma|$.*

This will allow us to bound the lengths of geodesics under K-quasiconformal maps in terms of their pre-images. We also recall the following classical result:

Proposition 2.6. *There exists a function $\delta(K) > 0$ such that $\delta(K) \to 0$ as $K \to 1$, which satisfies the following. Let $f : S_1 \to S_2$ be a K-quasiconformal map between two hyperbolic Riemann surfaces and suppose that γ is a geodesic on S_1. Then $f(\gamma)$ is contained in a $\delta(K)$-neighborhood of the unique geodesic on S_2 homotopic to $f(\gamma)$.*

That is, the image of a geodesic under a K-quasiconformal map is contained within a collar of a geodesic, the width of which tends to 0 as K tends to 1. Finally, Proposition 1.16 from [7] gives us the following:

Proposition 2.7. *Let S be a closed Riemann surface of genus $g \geq 2$ equipped with its hyperbolic metric. Then the shortest curve c on S has length $|c| \leq 2\log(4g-2)$.*

These two facts will allow us to prove the main proposition in the next section.

3 Existence of a Suitable $f \in \mathcal{F}$

Let S be a closed hyperbolic Riemann surface with shortest geodesic c. Suppose that S is K-quasiconformally homogeneous with transitive family of K-quasiconformal maps $\mathcal{F}$.

In [2], a lower bound was obtained for the dilatation of pseudo-Anosov maps. Recall the condition in Lemmas 2.1 and 2.2, that we have some map f such that $[c] \neq [f(c)]$. Pseudo-Anosov maps are always homotopically nontrivial, so the existence of such an f is known a priori. For general K-quasiconformal homeomorphisms, this is not necessarily the case, but our proof of Theorem 1.2 makes use of the aforementioned lemmas with the curve c, as well as a nearby geodesic. Thus, we must show that there exists a map $f \in \mathcal{F}$ that sends both c and a neighboring geodesic to curves not homotopic to their preimages for surfaces with sufficiently small K. We phrase this as follows:

Proposition 3.1. *There exists a universal constant $K_0 > 1$ such that if S is a K_0-quasiconformally homogeneous closed Riemann surface of genus $g \geq 2$, and $\mathcal{F}$ is a transitive family of K_0-quasiconformal homeomorphisms of S, then if c is the shortest geodesic on S and d another geodesic on S whose length is at most $2|c|$ and such that the distance between c and d on S is at most $\frac{1}{|c|}$, there exists $f \in \mathcal{F}$ such that $[c] \neq [f(c)]$ and $[d] \neq [f(d)]$.*

We will prove this in three parts. First, we show that surfaces lacking a $f \in \mathcal{F}$ such that $[f(c)] \neq [c]$ and $[f(d)] \neq [d]$ must be contained in the union of small neighborhoods of c and d. Then, we will exhibit a bound on the area of such a surface. Finally, we show that for sufficiently small K this bound cannot hold.

Claim 3.2. *Let S be as in Proposition 3.1. If for all $f \in \mathcal{F}$, we have that f fixes at least one of $[c]$ or $[d]$, then S is contained in the union of a δ-neighborhood of c and a δ-neighborhood of d, where $\delta = \delta(K)$ depends on K and $\delta \to 0$ as $K \to 1$.*

Proof. By hypothesis, we can choose some $x \in S$ such that x is in a $\frac{1}{|c|}$-neighborhood of both c and d. Since $\mathcal{F}$ is transitive, for any point $y \in S$ we can find a map $f \in \mathcal{F}$ such that $f(x) = y$. By Proposition 2.6, we know that each $f \in \mathcal{F}$ sends any point on c or d to a point contained in a $\delta^*(K)$-neighborhood of a geodesic, where $\delta^* \to 0$ as $K \to 1$. By continuity, we know that each image of x will be in a slightly larger neighborhood of a geodesic, say a $\delta(K)$-neighborhood. Notice that as $|c|$ increases (as $K \to 1$

by Lemma 2.4), we have $\delta \to \delta^*$ because c and d are separated by a distance $\frac{1}{|c|}$.

Now, if all maps $f \in \mathcal{F}$ fix at least one of $[c]$ or $[d]$, the image of x must remain in a δ-neighborhood of at least one of these curves. It follows that S is contained in the union of a δ-neighborhood of c and a δ-neighborhood of d, where δ depends only on K. By Proposition 2.6, together with Lemma 2.4, we can make δ arbitrarily small by sending $K \to 1$. These neighborhoods are collars around the curves c and d of total width 2δ, and as remarked above, sending $K \to 1$ will send $\delta \to 0$. This completes the proof of the claim. □

In the rest of the proof, we show that when δ is small, S cannot be contained in these two $\delta(K)$-neighborhoods of c and d.

Claim 3.3. *Let S be as above. Then:*

$$\text{Area}(S) < 2\pi(\frac{3\log(4g-2)}{\delta} + 2)(\cosh(2\delta) - 1).$$

Proof. Since S is contained in small neighborhoods of the two curves c and d, we will bound the area of the neighborhoods of each curve from above as follows. Each collar can be covered by hyperbolic disks (2-balls) of radius 2δ whose centers lie on the main curve (See Figure 2). We arrange them such that each disk is separated by a distance of 2δ from the adjacent disks. That gives a total of less than $\frac{|c|}{2\delta} + 1$ disks (taking the smallest integer greater than $\frac{|c|}{2\delta}$) for curve c, and $\frac{|d|}{2\delta} + 1$ disks for d. One disk for each of c and d will be less than 2δ away from one of its neighbors if $|c|$ or $|d|$ is not an integer multiple of 2δ.

In order to show that the disks cover the entire collar, we must show that the height h of the hyperbolic right triangle (i.e. half of the width of the area covered by the disks) in Figure 2 is at least δ. In that case, the disks will cover a collar around their respective curve of total width at least 2δ (since the disks are convex), thus covering the δ-neighborhood of the curve. This follows from the hyperbolic Pythagorean theorem, which gives

$$\cosh(2\delta) = \cosh(\delta)\cosh(h).$$

Indeed, supposing otherwise and applying the appropriate identities, we have

$$h < \delta \Rightarrow \cosh(h) < \cosh(\delta) \Rightarrow \frac{\cosh(2\delta)}{\cosh(\delta)} < \cosh(\delta)$$

$$\Rightarrow 2\cosh^2(\delta) - 1 < \cosh^2(\delta) \Rightarrow \cosh^2(\delta) < 1,$$

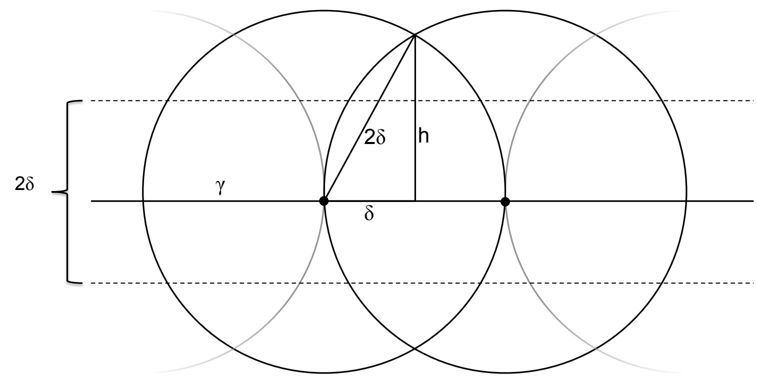

Figure 2: Covering the δ-neighborhood of the curve γ with disks of radius 2δ. The hyperbolic right triangle has base δ (half the separation between the disks) and height $h \geq \delta$. The hypotenuse is the radius of a disk. We do this for curves c and d

contradicting the fact that $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$. Thus, the disks cover the δ-neighborhoods of c and d, whence they cover S.

Recall that the area of a hyperbolic disk of radius r is $2\pi(\cosh r - 1)$. Since our collection of disks bounds the area of S from above, we have:

$$\text{Area}(S) < (\frac{|c|}{2\delta} + 1)2\pi(\cosh(2\delta) - 1) + (\frac{|d|}{2\delta} + 1)2\pi(\cosh(2\delta) - 1) \tag{2}$$

where $(\frac{|c|}{2\delta} + 1)$ and $(\frac{|d|}{2\delta} + 1)$ bound the number of disks from above, and $2\pi(\cosh(2\delta) - 1)$ is the area of each disk. Since by hypothesis we have that $|d| \leq 2|c|$, we can rewrite this as:

$$\text{Area}(S) < (\frac{3|c|}{2\delta} + 2)2\pi(\cosh 2\delta - 1) \tag{3}$$

From Proposition 2.7 we also have that $|c| \leq 2\log(4g - 2)$. Together with (3), this gives:

$$\text{Area}(S) < 2\pi(\frac{3\log(4g-2)}{\delta} + 2)(\cosh 2\delta - 1), \tag{4}$$

as desired. $\square$

Armed with the inequality (4), we can proceed to the final proof of Proposition 3.1.

Proposition 3.1. Let S be as above. We need to show that there exists a universal constant $K_0 > 1$ such that if $K \leq K_0$, then for some $f \in \mathcal{F}$, we have both $[f(c)] \neq [c]$ and $[f(d)] \neq [d]$. We show that K is bounded from below by some $K_0 > 1$ for surfaces with families in which no such f exists. Recall that the area of a hyperbolic surface S of genus $g \geq 2$ is given by:

$$\text{Area}(S) = 4\pi(g-1). \tag{5}$$

Now, from the previous claim we have an upper bound for the area of our surface S in terms of the genus g and $\delta = \delta(K)$. Combining (4) and (5), we have:

$$4\pi(g-1) < 2\pi(\frac{3\log(4g-2)}{\delta} + 2)(\cosh(2\delta) - 1). \tag{6}$$

This inequality follows from the upper bound on the area from Claim 3.3, the area of a hyperbolic surface from (5), and the upper bound on the lengths of c from Proposition (2.7). Simplifying, we obtain:

$$\frac{2(g-1)}{\log(4g-2)} < (\frac{3}{\delta} + \frac{2}{\log(4g-2)})(\cosh(2\delta)-1) < (\frac{3}{\delta}+2)(\cosh(2\delta)-1). \tag{7}$$

We have $g \geq 2$, and so (7) gives:

$$\frac{2}{\log(6)} < (\frac{3}{\delta} + 2)(\cosh(2\delta) - 1). \tag{8}$$

Notice that for the upper bound in (8), we have

$$\lim_{\delta \to 0}(\frac{3}{\delta} + 2)(\cosh(2\delta) - 1) = 0. \tag{9}$$

Now, (8) and (9) show that there exists a uniform lower bound on δ. By Lemma 2.4 and Proposition 2.6, we can choose $K > 1$ such that δ becomes arbitrarily small, which sends the right-hand side of (8) to 0. Thus, there must be a universal lower bound $K_0 > 1$ on K. If the transitive family $\mathcal{F}$ does not include maps that send c to non-homotopic curves, then $K > K_0$.

Thus for $1 < K \leq K_0$, the transitive family $\mathcal{F}$ must include a map that sends both c and d to a non-homotopic curve.

□

Using Proposition 3.1, will now prove Theorem 1.2.

4 Proof of Theorem 1.2

Proof. Let S be a closed hyperbolic Riemann surface, and suppose S is Torelli-K-quasiconformally homogeneous with family of homeomorphisms $\mathcal{F}$. Suppose $1 < K \leq K_0$ from Proposition 3.1. Let c be a simple closed geodesic on S of minimal length. Using Proposition 3.1, choose some $f \in \mathcal{F}$ such that $[c] \neq [f(c)]$ and f is also homotopically non-trivial on any geodesic in a $\frac{1}{|c|}$-neighborhood of c.

We have the following two cases: (1) $i(c, f(c)) \geq 4$ or $i(c, f^2(c)) \geq 4$, or (2) c is nonseparating and $i(c, f(c))$ and $i(c, f^2(c))$ are both less than 4. Notice that these cover all possibilities: if c is separating, then Lemma 2.1 gives us that $i(c, f(c)) \geq 4$. If c is nonseparating, then either one of $i(c, f(c))$ or $i(c, f^2(c))$ is greater than 4, which is case 1, and otherwise we have case 2.

Case 1: Let h be either f or f^2, where $i(c, h(c)) \geq 4$. Since $i(c, h(c)) \geq 4$, any member of the homotopy class $[h(c)]$ will have at least 4 intersections with c. In particular, let c' be the geodesic homotopic to $h(c)$. The intersection points $c \cap c'$ cut c and c' into arcs. Since there are at least 4 such points, there is an arc a of c' which satisfies

$$|a| \leq \frac{|c'|}{4} \leq \frac{K^2|c|}{4}. \tag{10}$$

The second inequality follows from Lemma 2.5, with K^2 since h is possibly f^2, and f^2 is a K^2-quasiconformal homeomorphism. The endpoints of a cut c into two arcs, one of which, say b, has length $|b| \leq |c|/2$. The union $a \cup b$ is a simple closed curve. It must be homotopically nontrivial since otherwise we could, by homotopy, reduce the number of intersections of c and c' below $i(c, c')$. Now, recall c is the shortest closed geodesic, so $|c| \leq |a \cup b|$. Then we have:

$$|c| \leq |a| + |b| \leq \frac{K^2|c|}{4} + \frac{|c|}{2} \Rightarrow 1 \leq \frac{K^2}{4} + \frac{1}{2} \tag{11}$$

Thus, $2 \leq K^2 \Rightarrow K \geq \sqrt{2}$.

Case 2: By Lemma 2.2, either $i(c, f(c)) = 2$ or $i(c, f^2(c)) = 2$. Let h be either f or f^2, where $i(c, h(c)) = 2$. Now, let c' be the geodesic homotopic to $h(c)$; we still have $i(c, c') = 2$. Let d and d' be the separating curves from Lemma 2.3 with c and c' as the homologous pair (since f is Torelli). Alternatively, the intersection points $c \cap c'$ define two arcs of c, say a and

a', and two arcs of c', say b and b' as in Figure 1. The curves d and d' are then $d \sim a \cup b$ and $d' \sim a' \cup b'$. Now,

$$|d| + |d'| = |a| + |b| + |a'| + |b'| = |c| + |c'| \leq |c| + K^2|c| \tag{12}$$

by Lemma 2.5 and since h may be f^2, which is a K^2-quasiconformal homeomorphism. It follows that at least one of d and d', say d, has length bounded above by half of $|c| + |c'|$:

$$|d| \leq \frac{|c| + K^2|c|}{2} \tag{13}$$

We now consider d_1, the geodesic homotopic to d, which is a separating curve, and the geodesic e_1 homotopic to e. We have $|d_1| \leq |d|$ and $|e_1| \leq |e|$. To continue, we require the following lemma, which will be proven at the end:

Lemma 4.1. *For sufficiently small K, the curves c and d_1 are within a distance of $\frac{1}{|c|}$ from each other.*

Now, suppose $K > 1$ is small enough so Lemma 4.1 can be used. Applying the lemma and Proposition 3.1, we see that that $[f(d_1)] \neq [d_1]$. We can now apply Lemma 2.1, so that $i(d_1, f(d_1))$ is at least 4. As in Case 1, if $\tilde{a}$ is the shortest arc of the geodesic homotopic to $f(d_1)$ cut off by d_1 and $\tilde{b}$ is the shortest arc of d_1 cut off by $\tilde{a}$, then

$$|\tilde{a} \cup \tilde{b}| \leq |d_1|(\frac{K}{4} + \frac{1}{2}). \tag{14}$$

Note that we can use K instead of possibly K^2 since d is separating, so Lemma 2.1 applies with f. Recall that we also have

$$|c| \leq |\tilde{a} \cup \tilde{b}|. \tag{15}$$

Combining (13), (14), (15), and the fact that $|d_1| \leq |d|$, we see that

$$|c| \leq (\frac{K}{4} + \frac{1}{2})(\frac{|c| + K^2|c|}{2}) \tag{16}$$

Which then gives

$$1 \leq (\frac{K}{4} + \frac{1}{2})(\frac{1 + K^2}{2}) \Rightarrow K^3 + 2K^2 + K - 6 \geq 0. \tag{17}$$

The cubic polynomial in K has one real root, and so

$$K \geq -\frac{2}{3} + \frac{1}{3}\sqrt[3]{82 - 9\sqrt{83}} + \frac{1}{3}\sqrt[3]{82 + 9\sqrt{83}} \approx 1.218 \tag{18}$$

approximated from below.

Now, since these bounds are independent of genus, we have that $K > 1.218$. We can then see that for some $K_T > 1$, any such surface S cannot be Torelli-K-quasiconformally homogeneous with $K \leq K_T$. Thus we have a universal constant $K_T > 1$ such that any Torelli-K-quasiconformally homogeneous closed hyperbolic Riemann surface of genus $g \geq 2$ must have $K \geq K_T > 1$. This completes the proof.

□

Proof of Lemma 4.1. First, consider the pair of pants defined by curves c, d_1, and e_1, as in Figure 1. Notice that as K approaches 1, we eventually have:

$$|c| \leq |d_1| \leq \frac{3|c|}{2}, |c| \leq |e_1| \leq \frac{3|c|}{2}. \tag{19}$$

This follows from (13), which can also apply with the same bounds to the curve e_1, and the fact that c is the shortest geodesic on S.

Consider now the right-angled hyperbolic hexagon formed by inserting perpendiculars connecting each of c, d_1, and e_1, and cutting off a fundamental hexagon from this pair of pants. Then there are three alternating sides of length $\frac{|c|}{2}$, $\frac{|d_1|}{2}$, and $\frac{|e_1|}{2}$. By Lemma 2.4 we also have that as $K \to 1$, $|c| \to \infty$. Recall from [8] the Law of Cosines for right-angled hyperbolic hexagons:

$$\cosh(z') = \coth(x)\coth(y) + \frac{\cosh(z)}{\sinh(x)\sinh(y)}, \tag{20}$$

where x, y, and z are the lengths of alternate sides of the hexagon, and z' is the length of the side opposite the side of length z. Let x, y, and z correspond to the sides of length $\frac{|c|}{2}$, $\frac{|d_1|}{2}$, and $\frac{|e_1|}{2}$ respectively, and so z' is the distance between curves c and d_1. We wish to show that $z' \to 0$ faster than $\frac{1}{|c|} \to 0$ as $K \to 1$. Notice from (19) that, as $|c| \to \infty$, x, y, and z all increase within a factor of $\frac{3}{2}$ from each other. Thus, the $\coth(x)\coth(y)$ term proceeds exponentially to 1, and the $\frac{\cosh(z)}{\sinh(x)\sinh(y)}$ term exponentially approaches 0.

Now, the right-hand side of (20) proceeds to unity, and so $\cosh(z') \to 1$. We can then see that z', the distance between curves c and d_1, approaches zero exponentially in $|c|$ as $|c|$ increases. Therefore as $K \to 1$, the curves c

and d_1 will approach each other exponentially in $|c|$, and so for sufficiently small K, they will be closer than $\frac{1}{|c|}$, as desired.

$\square$

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